

AMENABLE SEMIGROUPS OF NONEXPANSIVE MAPPINGS ON WEAKLY COMPACT CONVEX SETS

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*Dedicated to Professor Tomás Domínguez Benavides
on the occasion of his 65th birthday*

ABSTRACT. We show a few fixed point theorems for semigroups acting on weakly compact convex subsets of Banach spaces when $LUC(S)$, $AP(S)$, $WAP(S)$ or $WAP(S) \cap LUC(S)$ have a left invariant mean. In particular, we give a characterization of semitopological semigroups that have a left invariant mean on the space of weakly almost periodic functions in terms of a fixed point property for nonexpansive mappings. It answers, in the case of Banach spaces, Question 4 of [A.T.-M. Lau, Y. Zhang, J. Funct. Anal. 263 (2012), 2949–2977] in affirmative. We also extend in Banach spaces the fixed point theorem of R. Hsu from left reversible discrete semigroups to left amenable semitopological semigroups.

1. INTRODUCTION

A well-known characterization of amenable semigroups is given by the following Day fixed point theorem: a semigroup S is (left) amenable if and only if whenever S acts affinely (from the left) on a nonempty compact convex subset K of a locally convex space, there is a common fixed point of S in K . This characterization was extended to topological groups by N. Rickert and to semitopological semigroups by T. Mitchell.

A natural question arises whether a similar characterization can be given in terms of nonexpansive mappings. The first results in this direction were given by W. Takahashi, T. Mitchell and A. T.-M. Lau. The following fixed point property is a corollary of [6, Theorem 4.1] (where a more general case of locally convex spaces was considered).

Theorem 1.1. *Let S be a semitopological semigroup. Then $AP(S)$, the space of continuous almost periodic functions on S , has a LIM (left invariant mean) if and only if S has the following fixed point property:*

(D): *Whenever the action of S on a compact convex subset K of a Banach space is separately continuous and nonexpansive, there is a common fixed point of S in K .*

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This article is motivated by the recent papers [10, 11], where the similar results were obtained for $WAP(S)$, the space of continuous weakly almost periodic functions on S , and $LUC(S)$, the space of left uniformly continuous functions on S , under the additional assumption that S is separable (or at least the set K on which S acts is separable). We show that it is possible to omit the separability assumption when S acts on a weakly compact convex subset of a Banach space. Thus we give a complete characterization of a sentence “ $WAP(S)$ has a LIM” in terms of a fixed point property for nonexpansive mappings that answers, in the case of Banach spaces, Question 4 in [11] in affirmative. A similar method allows us to strengthen Theorem 1.1 above. Moreover, we obtain the fixed point theorems for semigroups of nonexpansive mappings when $LUC(S)$ or $WAP(S) \cap LUC(S)$ have a left invariant mean. In particular, the fixed point theorem of Hsu [5] (see also [11, Theorem 3.10]) is extended from left reversible discrete semigroups to left amenable semitopological semigroups acting on weakly compact convex subsets of Banach spaces.

2. PRELIMINARIES

Let S be a semitopological semigroup, i.e., a semigroup with a Hausdorff topology such that the mappings $S \ni s \rightarrow ts$ and $S \ni s \rightarrow st$ are continuous for each $t \in S$. Notice that every semigroup can be equipped with the discrete topology and then it is called a discrete semigroup. Let $\ell^\infty(S)$ be the Banach space of bounded complex-valued functions on S with the supremum norm. For $s \in S$ and $f \in \ell^\infty(S)$, we define the left and right translations of f in $\ell^\infty(S)$ by

$$l_s f(t) = f(st) \text{ and } r_s f(t) = f(st)$$

for every $t \in S$. Let X be a closed linear subspace of $\ell^\infty(S)$ containing constants and invariant under translations, i.e., $l_s(X) \subset X$ and $r_s(X) \subset X$. Then a linear functional $m \in X^*$ is called a left invariant mean on X (LIM, for short), if $\|m\| = m(1) = 1$ and

$$m(l_s f) = m(f)$$

for each $s \in S$ and $f \in X$. Similarly, we can define a right invariant mean.

Denote by $C(S)$ the closed subalgebra of $\ell^\infty(S)$ consisting of continuous functions and let $LUC(S)$ be the space of left uniformly continuous functions on S , i.e., all $f \in C(S)$ such that the mapping $S \ni s \rightarrow l_s f$ from S to $C(S)$ is continuous when $C(S)$ has the sup norm topology. A semigroup S is called left amenable if there exists a left invariant mean on $LUC(S)$. Left uniformly continuous functions are often called in the literature “right uniformly continuous” and vice-versa.

Two other subspaces of $C(S)$ are very important in this context. A bounded continuous function f on S is called almost periodic if $\{l_s f : s \in S\}$ (equivalently, $\{r_s f : s \in S\}$) is relatively compact in the norm topology of $C(S)$. A bounded continuous function f on S is called weakly almost periodic if $\{l_s f : s \in S\}$ (equivalently, $\{r_s f : s \in S\}$) is relatively compact in the weak topology of $C(S)$. The space of almost periodic (resp., weakly

almost periodic) functions on S is denoted by $AP(S)$ (resp., $WAP(S)$). In general,

$$AP(S) \subset LUC(S) \text{ and } AP(S) \subset WAP(S),$$

and if S is discrete, then

$$AP(S) \subset WAP(S) \subset LUC(S) = \ell^\infty(S).$$

Let K be a topological space. A semigroup S is said to act on K (from the left) if there is a map

$$\psi : S \times K \rightarrow K$$

such that

$$\psi(s, \psi(s', x)) = \psi(ss', x)$$

for all $s, s' \in S$ and $x \in K$. We write $\psi(s, x) = s \cdot x = T_s x$. The action is said to be separately continuous if it is continuous in each of the variables when the other is fixed. If K is a subset of a Banach space E , the action \cdot is called nonexpansive if

$$\|s \cdot x - s \cdot y\| \leq \|x - y\|$$

for every $x, y \in K$ and $s \in S$. Given an action \cdot of S , an element $x \in K$ is called a common fixed point for S in K if $s \cdot x = x$ for every $s \in S$.

If X is a closed linear subspace of $\ell^\infty(S)$ invariant under translations then X is said to be (left) introverted if for any $\varphi \in X^*$ and $f \in X$, the function $h(s) = \varphi(l_s f)$, $s \in S$, belongs to X . It is known that $AP(S)$, $WAP(S)$ and $LUC(S)$ are introverted subspaces of $\ell^\infty(S)$ (see, e.g., [14, Prop. 2.11]). Since we work in Banach spaces rather than in locally convex topological spaces we shall need the following theorem of Granirer and Lau (compare [3, Theorem 1], [11, Remark 6.4], and, in the case of locally compact groups, [14, Theorem 2.13]).

Theorem 2.1. *Let X be an introverted subspace of $\ell^\infty(S)$ with $1 \in X$. Then X has a LIM if and only if $\overline{\text{co}}^p\{r_s f : s \in S\}$, the pointwise closure of the convex hull of the right orbit of f , contains a constant function for every $f \in X$.*

Furthermore, X (containing constants) is introverted if and only $\overline{\text{co}}^p\{r_s f : s \in S\} \subset X$ (see [3, Lemma 2]).

Let μ be a (locally finite, positive) Radon measure on a topological space K , i.e., a Borel measure which is inner regular: $\mu(A) = \sup\{\mu(C) : C \subset A, C \text{ compact}\}$ for any measurable set A . Recall that the support of μ is defined as the complement of the set of points that have neighborhoods of measure 0. In general, the support may be empty but if K is (at least) locally compact and μ a finite Radon measure, then $\mu(\text{supp}(\mu)) = \mu(K)$. The crucial observation for this paper is the following theorem of Grothendieck [4].

Theorem 2.2. *Every finite Radon measure on a weakly compact set in a Banach space has a norm-separable support.*

Proof. We sketch the proof following Todorčević [17, Theorem 9.1], see, e.g., [12, Theorem 4.3] for two detailed proofs. Since $S = \text{supp}(\mu)$ satisfies the countable chain condition and is weakly compact, it follows from [15, Theorem 1.4] that every weakly compact subset of $C(S)$ is separable. Hence $C(S)$ is separable as a weakly compactly generated space by [1, Prop. 1]. It follows that S is also separable. \square

3. FIXED POINT THEOREMS

Let S be a semitopological semigroup. An action $\cdot : S \times K \rightarrow K$ on a Hausdorff space K is called equicontinuous if the family of functions $\{K \ni x \rightarrow s \cdot x \in K\}_{s \in S}$ is equicontinuous in the usual sense. It is called quasi-equicontinuous if \bar{S}^p , the closure of $\{K \ni x \rightarrow s \cdot x \in K\}_{s \in S}$ in K^K with the product topology, consists of only continuous mappings. If K is a subset of a Banach space E , then the action on K is called weakly equicontinuous (resp., weakly quasi-equicontinuous) if it is equicontinuous (resp., quasi-equicontinuous) when K is equipped with the weak topology of E .

The aim of this section is to prove the following fixed point properties.

Theorem 3.1. *WAP(S) has a LIM if and only if S has the following fixed point property:*

(F): *Whenever S acts on a weakly compact convex subset K of a Banach space and the action is weakly separately continuous (i.e., separately continuous when K is equipped with the weak topology), weakly quasi-equicontinuous and nonexpansive, then K contains a common fixed point for S .*

In the case of Banach spaces, the above theorem drops the separability assumption from [10, Theorem 3.4] and answers in affirmative [11, Question 4] in that case.

Theorem 3.2. *AP(S) has a LIM if and only if S has the following fixed point property:*

(E): *Whenever S acts on a weakly compact convex subset K of a Banach space and the action is weakly separately continuous, weakly equicontinuous and nonexpansive, then K contains a common fixed point for S .*

It removes the separability assumption from [10, Theorem 3.6] in the case of Banach spaces and thus strengthen Theorem 1.1 which was alluded to in the introduction.

In addition to the above characterization theorems, we obtain the following fixed point theorems.

Theorem 3.3. *If $WAP(S) \cap LUC(S)$ has a LIM, then S has the following fixed point property:*

(F*): *Whenever S acts on a weakly compact convex subset K of a Banach space and the action is (jointly) weakly continuous, weakly quasi-equicontinuous and nonexpansive, then K contains a common fixed point for S .*

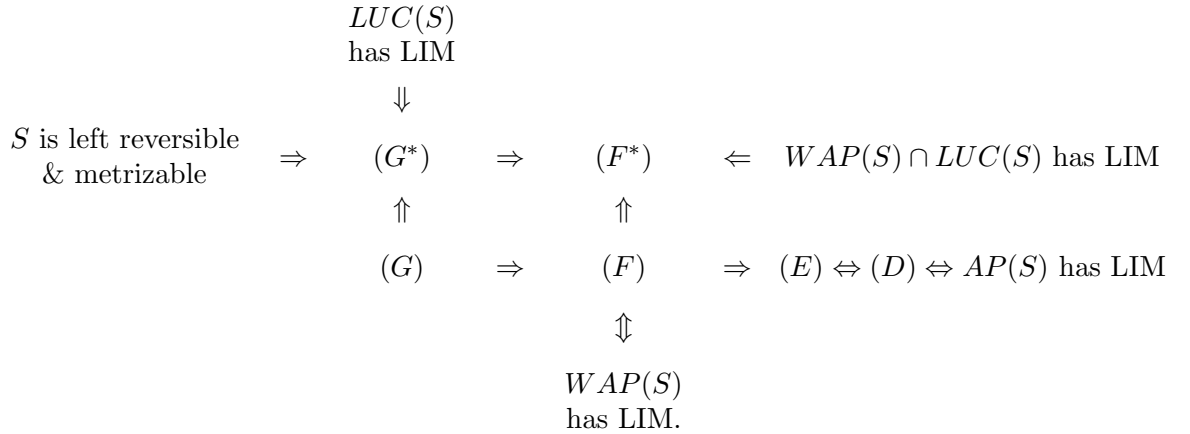
It drops the separability assumption from the part of [10, Theorem 5.1]. It would be interesting to prove also the reverse implication as it was done there in the case of locally convex spaces.

Theorem 3.4. *If $LUC(S)$ has a LIM, then S has the following fixed point property:*

(G^): Whenever S acts on a weakly compact convex subset K of a Banach space and the action is (jointly) weakly continuous and nonexpansive, then K contains a common fixed point for S .*

Note that Hsu [5] proved that in the more general case of locally convex spaces, left reversible discrete semigroups have property (G^*) and Lau and Zhang [10, Theorem 5.4] generalized Hsu's result to left reversible, metrizable semitopological semigroup. Theorem 3.4 extends it also to left amenable semigroups (in Banach spaces).

The following diagram summarizes the known relations among the fixed point properties of semitopological semigroups acting on weakly compact convex subsets of Banach spaces (compare the diagram on p. 2553 in [10]):



We begin with a general lemma which is patterned after [6, Theorem 4.1] and [8, Lemma 5.1]. We will denote by $C(K)$ the space of weakly continuous complex-valued functions defined on a weakly compact set K .

Lemma 3.5. *Let S be a semitopological semigroup and X a closed linear subspace of $\ell^\infty(S)$ containing constants and invariant under translations. Suppose that S acts on a weakly compact subset K of a Banach space E so that the action is weakly separately continuous and there exists $y \in K$ such that for every $f \in C(K)$, the function $S \ni s \rightarrow f_y(s) = f(s \cdot y)$ belongs to X . If X has a left invariant mean, then there exists a nonempty weakly compact and norm-separable subset K_0 of K such that $sK_0 = \{s \cdot x : x \in K_0\} = K_0$ for every $s \in S$.*

Proof. Let m be a left invariant mean on X and define a positive functional Φ on $C(K)$ by

$$\Phi(f) = m(f_y)$$

for every $f \in C(K)$. Notice that Φ is well-defined since, by assumption, $f_y \in X$, and $\|\Phi\| = 1$. Define

$${}_t f(x) = f(t \cdot x), \quad x \in K,$$

for every $t \in S$ and $f \in C(K)$. Then ${}_t f : K \rightarrow \mathbb{C}$ is weakly continuous since the action \cdot is weakly separately continuous and

$$\Phi(f) = \Phi({}_t f).$$

Let μ be the probability Radon measure on K corresponding to Φ . Then $\mu(A) = \mu(s^{-1}A)$ for every Borel subset A of K (with the weak topology) and $s \in S$, where as usual, $s^{-1}A = \{x \in K : s \cdot x \in A\}$. Define $K_0 = \text{supp}(\mu)$ and notice that $\mu(s^{-1}K_0) = \mu(K_0) = 1$. Hence $K_0 \subset s^{-1}K_0$ since $s^{-1}K_0$ is weakly closed. Similarly, $\mu(sK_0) = \mu(s^{-1}(sK_0)) = \mu(K_0) = 1$ and consequently $K_0 \subset sK_0$ (notice that sK_0 is weakly compact). Thus $sK_0 = K_0$ for every $s \in S$. Moreover, K_0 is nonempty, weakly compact and separable by Theorem 2.2. \square

We can now state a general fixed point theorem for a semitopological semigroup S . The proof follows [8, Theorem 5.3]. We sketch it for the convenience of the reader.

Theorem 3.6. *Let S be a semitopological semigroup and X a closed linear subspace of $\ell^\infty(S)$ containing constants and invariant under translations. Suppose that S acts on a weakly compact subset K of a Banach space E so that the action is weakly separately continuous, nonexpansive and the function $S \ni s \rightarrow f_y(s) = f(s \cdot y)$ belongs to X for every $y \in K$ and every $f \in C(K)$. If X has a left invariant mean then there is a common fixed point of S in K .*

Proof. By Kuratowski-Zorn's lemma, there exists a nonempty minimal weakly compact and convex subset C of K which is invariant under S . Let F be a nonempty minimal weakly compact subset of C which is invariant under S . Fix $y \in F$ and notice that if $f \in C(F)$, then $f_y = \bar{f}_y \in X$, where $\bar{f} : K \rightarrow \mathbb{C}$ is an extension of f to the whole set K . Applying Lemma 3.5 there exists a weakly compact and norm-separable subset K_0 of F such that $sK_0 = K_0$ for every $s \in S$. From minimality of F , $K_0 = F$ is separable and $\{s \cdot x : s \in S\}$ is weakly dense in F for every $x \in F$. Thus F is norm-compact by [8, Lemma 5.2] (which in turn follows the ideas of Hsu [5] related to “fragmentability”, see [13]). Suppose that $r = \text{diam } F > 0$. Then by [2, Lemma 1], there is $u \in \overline{\text{co}} F \subset C$ such that $r_0 = \sup\{\|u - y\| : y \in F\} < r$. Let

$$C_0 = \{x \in C : \|x - y\| \leq r_0 \text{ for all } y \in F\}.$$

Then $u \in C_0$ and C_0 is a weakly compact convex proper subset of C . Since the action is nonexpansive and $sF = F$ it follows that $sC_0 \subset C_0$ for all $s \in S$ which contradicts the minimality of C . Thus $\text{diam } F = 0$ and F consists of a single point which is a common fixed point of S in K . \square

Theorem 3.6, together with Theorem 2.1 and some results from [10, 11], allows proving theorems stated at the beginning of this section in a compact way. The remainder of this section will be devoted to their proofs.

Proof of Theorem 3.1. Assume that $WAP(S)$ has a LIM. Since the action is weakly separately continuous and weakly quasi-equicontinuous, it follows from [10, Lemma 3.2] that the function $f_y(s) = f(s \cdot y)$, $s \in S$, belongs to $WAP(S)$ for every $f \in C(K)$ and $y \in K$. Thus, the assumptions of Theorem 3.6 are satisfied with $X = WAP(S)$ and we obtain a common fixed point of S in K .

To prove the reverse implication, we follow [11, Prop. 6.5]. Assume that the semigroup S is a monoid. It does not lose the generality, see [11, Lemma 6.3]. Fix $f \in WAP(S)$ and let $K_f = \overline{\text{co}}\{r_s f : s \in S\}$ be the norm closure of the convex hull of the right orbit of f . Then K_f is weakly compact, and notice that the mapping $(s, g) \rightarrow r_s g$, $s \in S$, $g \in K_f$, defines a left action of S on K_f . Indeed,

$$r_{s'} r_s g(t) = r_s g(ts') = g(ts's) = r_{s's} g(t)$$

for every $s, s', t \in S$, $g \in K_f$, and consequently,

$$r_s(\alpha_1 r_{s_1} f + \dots + \alpha_n r_{s_n} f) = \alpha_1 r_{ss_1} f + \dots + \alpha_n r_{ss_n} f$$

so that $r_{s'}(\text{co}\{r_s f : s \in S\}) \subset \text{co}\{r_s f : s \in S\}$ for every $s' \in S$. Furthermore, the action is affine and nonexpansive:

$$\sup_{t \in S} \|r_s g(t) - r_s h(t)\| = \sup_{t \in S} \|g(ts) - h(ts)\| \leq \sup_{t \in S} \|g(t) - h(t)\|.$$

Thus $r_s(K_f) \subset K_f$ and r_s is weakly continuous on K_f for every $s \in S$. To prove that the mapping $S \ni s \rightarrow r_s g \in K_f$ is also continuous for every fixed $g \in K_f$, choose a net $(s_\alpha) \subset S$ converging to s in the topology of S . Then

$$r_s g(t) = g(t \lim_{\alpha} s_\alpha) = \lim_{\alpha} g(ts_\alpha) = \lim_{\alpha} r_{s_\alpha} g(t), \quad t \in S,$$

and since K_f is weakly compact, the pointwise limit $\lim_{\alpha} r_{s_\alpha} g(\cdot)$ must agree with the weak limit $w\text{-}\lim_{\alpha} r_{s_\alpha} g$. It follows that the action $S \times K_f \ni (s, g) \rightarrow r_s g \in K_f$ is weakly separately continuous.

The next step is to show that this action is also weakly quasi-equicontinuous. We follow the argument in [10, Theorem 3.4]. Choose a net $(s_\alpha) \subset S$ such that $w\text{-}\lim_{\alpha} r_{s_\alpha} g = T(g)$ for each $g \in K_f$. We have to show that $T : K_f \rightarrow K_f$ is weakly continuous. Suppose on the contrary that there exists a net $(g_\beta) \subset K_f$ such that $w\text{-}\lim_{\beta} g_\beta = g \in K_f$ but $(T(g_\beta))_\beta$ does not converge weakly to $T(g)$. Then there exists $\varepsilon > 0$, a functional $\varphi \in E^*$ and a subnet of (g_β) (still denoted by (g_β)) such that $\text{Re}(\langle \varphi, T(g_\beta) - T(g) \rangle) > \varepsilon$ for all β . By Mazur's lemma, there is a net $(h_\lambda) \subset \text{co}(g_\beta)$ which converges in norm to g . But T is affine since all r_{s_α} are affine and hence $\text{Re}(\langle \varphi, T(h_\lambda) - T(g) \rangle) > \varepsilon$ for all λ . On the other hand,

$$\begin{aligned} |\langle \varphi, T(h_\lambda) - T(g) \rangle| &\leq \|\varphi\| \left\| w\text{-}\lim_{\alpha} r_{s_\alpha} h_\lambda - w\text{-}\lim_{\alpha} r_{s_\alpha} g \right\| \\ &\leq \|\varphi\| \limsup_{\alpha} \|r_{s_\alpha} h_\lambda - r_{s_\alpha} g\| \leq \|\varphi\| \|h_\lambda - g\| \xrightarrow{\lambda} 0, \end{aligned}$$

and we obtain a contradiction which shows that the action is weakly quasi-equicontinuous. By the fixed point property (F) , for every $f \in WAP(S)$ there exists $g \in K_f$ such that $r_s g = g$ for every $s \in S$. It follows that $g(ts) = g(t)$ for every $t \in S$ and taking $t = e$, the identity of S , we have

that $g(s) = g(e)$ is a constant function. Since $WAP(S)$ is introverted and $K_f = \overline{\text{co}}^p\{r_s f : s \in S\}$ contains a constant function for every $f \in WAP(S)$, the conclusion follows from Theorem 2.1. \square

Proof of Theorem 3.2. Assume that $AP(S)$ has a LIM. Since the action is weakly separately continuous and weakly equicontinuous, $f_y \in AP(S)$ for every $f \in C(K)$ and $y \in K$ by [6, Lemma 3.1]. Applying Theorem 3.6 with $X = AP(S)$ we get a common fixed point of S in K .

To prove the converse, assume that S is a monoid. Fix $f \in AP(S)$ and let $K_f = \overline{\text{co}}\{r_s f : s \in S\}$. Now K_f is compact and define, as before, a left action of S on K_f by

$$S \times K_f \ni (s, g) \rightarrow r_s g \in K_f.$$

The action is weakly separately continuous and nonexpansive, and hence weakly equicontinuous since the weak topology coincides with the norm topology on K_f . By property (E), for every $f \in AP(S)$ there exists $g \in K_f$ such that $r_s g = g$ for every $s \in S$. The rest of the proof runs as before. \square

Proof of Theorem 3.3. By assumption, the action is jointly weakly continuous and weakly quasi-equicontinuous, hence $f_y \in WAP(S)$ for every $f \in C(K)$ and $y \in K$ as in the proof of Theorem 3.1. Furthermore, $f_y \in LUC(S)$ by [8, Lemma 5.1] (joint weak continuity is enough here). Therefore, if $WAP(S) \cap LUC(S)$ has a LIM, we can apply Theorem 3.6 with $X = WAP(S) \cap LUC(S)$ to obtain a common fixed point of S in K . \square

Proof of Theorem 3.4. If the action is jointly weakly continuous then $f_y \in LUC(S)$ for every $f \in C(K)$ and $y \in K$ by [8, Lemma 5.1]. Since the action is also nonexpansive and $LUC(S)$ has a left invariant mean, the result follows from Theorem 3.6. \square

Recall that a bounded convex subset C of a Banach space has normal structure if $r(C) < \text{diam } C$, where $r(C) = \inf_{x \in C} \sup_{y \in C} \|x - y\|$ is the Chebyshev radius of C . Clearly, Theorem 3.2 shows that if $AP(S)$ has a left invariant mean then a semitopological semigroup S has the following property:

(E'): Whenever S acts on a weakly compact convex subset K of a Banach space with normal structure and the action is weakly separately continuous, weakly equicontinuous and nonexpansive, then K contains a common fixed point for S .

Notice that since norm-compact convex subsets of Banach spaces have normal structure, we can follow the proof of Theorem 3.2 and show that the reverse implication is also true. Thus property (E') is equivalent to property (E). A similar result was proved in [10] for separable semigroups in the case of locally convex spaces.

The results of this paper have a natural generalization to semigroups acting on weak-star compact convex subsets of a dual Banach space with the

Radon-Nikodym property and, more generally, on norm-fragmented spaces. This problem will be studied in a subsequent publication.

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